Efficient formulas for $\zeta(n)$

by
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Abstract

In the Ramanujan Notebooks, we can see an original formula.

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

Which inspired me to find similar ones much later with powers of π and $\zeta(n)$, $n > 1$. Here I present a new range of formulas of the same kind, but considerably more efficient for calculation. They apply to all values of n. These new formulas allow all Zeta function values to be evaluated with an efficiency of 3.85 and 5.46 decimal places per term. By exploiting some of these series we obtain an explicit formula for the n'th decimal place of $\zeta(3)$ and a new efficient formula for calculating ln(2). Other presentations allow you to calculate the nth decimal of the numbers $1/\pi^{2n+1}$.

Résumé

Dans les Ramanujan Notebooks, on peut apercevoir une formule originale.

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

Qui m'a inspiré pour en trouver d'autres semblables bien après avec les puissances de π et $\zeta(n)$, $n > 1$. Je présente ici une nouvelle gamme de formules du même genre mais nettement plus efficaces pour le calcul. Elles s'appliquent à toutes les valeurs de n. Ces nouvelles formules permettent d'évaluer les toutes les valeurs de la fonction Zeta avec une efficacité de 3.85 et 5.46 décimales par terme. En exploitant certaines de ces séries on obtient une formule explicite pour la n'ième décimale de $\zeta(3)$ et une nouvelle formule efficace pour le calcul de ln(2). D'autres présentées permettent de calculer la nième décimales des nombres $1/\pi^{2n+1}$.

Introduction

We go back in 2006 where I have found these formulas.

$$\zeta(5)=\frac{694}{204813}\pi^5-\frac{6280}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}+\frac{296}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{5\pi n}-1)}-\frac{1073}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{10\pi n}-1)}+\frac{37}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20\pi n}-1)}$$

$$\zeta(5)=\frac{11\pi^5\sqrt{3}}{5670}+2\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{3}\pi n}-1)}-\frac{33}{8}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{12}\pi n}-1)}+\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{48}\pi n}-1)}$$

$$\zeta(3)=\frac{13\pi^3\sqrt{3}}{45}+2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\sqrt{3}\pi n}-1)}-\frac{9}{2}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\sqrt{3}\pi n}-1)}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\sqrt{3}\pi n}-1)}$$

$$\zeta(5)=\frac{5\pi^5\sqrt{7}}{3906}+\frac{64}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{7}\pi n}-1)}+\frac{130}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{28}\pi n}-1)}-\frac{4}{31}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\sqrt{112}\pi n}-1)}$$

We notice a pattern in the exponents (the term with $e^{\pi n}$), the pattern is always (1, 2, 4) but it's not the only one. In the case of ζ(5) the pattern is more like (4, 5, 10, 20). In fact on close inspection the pattern (1, 2, 4) are the divisors of 4 and (4, 5, 10, 20) is a subset of the divisors of 20. Furthermore, the series for ζ(5) converges relatively quickly since each term gives 5.46 decimal places. The pattern (4, 5, 10, 20) is also repeated with all ζ(4n+1).

Indeed, the coefficients for n=5, 9, 13, 17, ... with the pattern (4, 5, 10, 20) persists.

$\zeta(n)$	Formula
$\zeta(5)$	$\frac{694}{204813}\pi^5-\frac{6280}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4n\pi}-1)}+\frac{296}{3251}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{5n\pi}-1)}-\frac{1073}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{10n\pi}-1)}+\frac{37}{6502}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20n\pi}-1)}$
$\zeta(9)$	$\frac{6118928}{2048182032863705}\pi^9-\frac{3908360}{32194573151}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{4n\pi}-1)}-\frac{15904}{1945731}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{5n\pi}-1)}+\frac{1011431}{676776}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{10n\pi}-1)}-\frac{497}{15565848}\sum_{n=1}^{\infty}\frac{1}{n^9(e^{20n\pi}-1)}$

$\zeta(13)$	$\frac{4131911428}{11996181573401025}\pi^{13}-\frac{2441359240}{1221199811}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{4n\pi}-1)}+\frac{1056896}{1221199811}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{5n\pi}-1)}-\frac{67121153}{39078393952}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{10n\pi}-1)}+\frac{8257}{39078393952}\sum_{n=1}^{\infty}\frac{1}{n^{13}(e^{20n\pi}-1)}$
$\zeta(17)$	$\frac{687182059214356}{19436286956855701770337 5}\pi^{17}-\frac{1525878246920}{762905503491}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{4n\pi}-1)}-\frac{66978304}{762905503491}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{5n\pi}-1)}+\frac{17180065793}{97651904446848}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{10n\pi}-1)}-\frac{130817}{97651904446848}\sum_{n=1}^{\infty}\frac{1}{n^{17}(e^{20n\pi}-1)}$

Using one of the formulae of L. Vepstas [11], other expressions can be found, there are an infinite number but the most efficient one that has been found for ζ(3) is;

$$\zeta(3)=\frac{17\pi^3}{310\sqrt{2}}-\frac{60}{31}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{4/\sqrt{2}n\pi}-1\right)}-\frac{4}{31}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{6/\sqrt{2}n\pi}-1\right)}+\frac{2}{31}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{12/\sqrt{2}n\pi}-1\right)}$$

Which gives 3.86 decimal places per term. It is also valid at values ζ(4n-1), the coefficients being different.

As for the formula with √7 and ζ(5), in the same way, it applies to values ζ(4n+1). So, the best found for ζ(5) and ζ(4n+1) still gives 5.46 decimal places per term. The efficiency increases with the exponent of n. To the best of my knowledge, these formulae are the simplest and most efficient known, even if we have to evaluate the value of π.

The formula with ζ(3) above is less efficient than the Amdeberhan and Zeilberger formulae but is simpler to program.

Another formula generator model is the following.

$$\zeta(2n+1) = A\,\pi^{2n+1} + B\sum_{n=1}^{\infty}\frac{1}{n^{2n+1}(e^{m\pi n}-1)} + C\sum_{n=1}^{\infty}\frac{1}{n^{2n+1}(e^{4/m\pi}-1)} \qquad 1$$

Where A, B et C are rationnals, for example with $\zeta(3), \zeta(5)$ et $\zeta(7)$ we get

$$\zeta(3) = \frac{1477}{19980}\pi^3 - \frac{2}{37}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{12n\pi}-1)} - \frac{72}{37}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{n\pi/3}-1)}$$

$$\zeta(5) = \frac{1493}{136350}\pi^5 + \frac{2}{9999}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{20n\pi}-1)} - \frac{20000}{9999}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{n\pi/5}-1)}$$

$$\zeta(7) = \frac{117799}{78189300}\pi^7 - \frac{2}{7529537}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{28n\pi}-1)} - \frac{15059072}{7529537}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{n\pi/7}-1)}$$

Valid for $\forall m \geq 3$.

So the 2nd term can converge very quickly if m ≫3, so much so that in practice from a certain precision it can be ignored. For example, with ζ(3), if m = 256, we obtain a precision of 353 decimal digits.

$$\zeta(3) \cong -\frac{268517377}{188755200}\pi^3 - \frac{2^{15}}{2^{14}+1}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{\frac{n\pi}{64}}-1\right)}$$

The coefficients (here C in the equation) are very easy to identify, B is more problematic but calculable all the same. It's a simple polynomial expression. The precision of the approximation increases geometrically, enough to derive a practical formula.

Its characteristics can be exploited since :

$$\zeta(3)_k = -\frac{1}{180}\frac{\pi^3(k^4+20k^2+16)+360k^3}{k(k^2+4)}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n4/k}-1)}$$

Approximate $\zeta(3)_k$ at k decimal digit.

In fact, with k = 1, 2, 3, ... we have

k	Value	Expression
1	1.<u>2</u>747010908032…	$\frac{37\pi^3}{900} - \frac{2}{5}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n4}-1)}$

2	1.$\underline{20}$39282759189…	$\frac{7\pi^3}{180}-\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n 2}-1)}$
3	1.$\underline{202}$1065689101…	$\frac{277\pi^3}{7020}-\frac{18}{13}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n 4/3}-1)}$
100	1.$\underline{2020569031595942853997}$…	$\frac{6262501}{11254500}\pi^3-\frac{5000}{2501}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{\frac{\pi}{25}}-1\right)}$
10^9	1.$\underline{2020569031595942853997}$…	$\frac{62500000000000001250000000000000001}{11250000000000000450000000000}\pi^3 - \frac{500000000000000000}{250000000000000001}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{\frac{\pi}{250000000}}-1\right)}$

Note that in the case of k=2, the formula is almost the same as Ramanujan's except for the factor 2 in the infinite sum. To get the k'th decimal place, all we need to do is put

$$[10^k(f(k))]\ \ mod\ 10$$

$f(k)$ being one of the formulae in the table.

They converge quite slowly but at least we have a compact and closed formula.

If we apply the same procedure for the other values of ζ(2n+1), the coefficients become quite large. For ζ(5) alone, I haven't managed to obtain the equivalent of a polynomial expression of degree 10, and it's even worse with ζ(7).

Take formula (1) again, it also works with exponent -1, here the constant is ln(2). The same phenomenon occurs: we have an identity with 2 terms, one of which converges very quickly and the other very slowly.

$$\ln(2) \approx A\,\pi - B\sum_{n=1}^{\infty}\frac{1}{n(e^{\left(\frac{\pi}{2^{n-2}}\right)}-1)}$$

The coefficients A and B are relatively simple to find, giving a compact formula

$$\ln(2) \approx \frac{\pi(2^{k-1}-2^{1-k})}{6(k-1)}-\frac{2}{k-1}\sum_{n=1}^{\infty}\frac{1}{n(e^{\left(\frac{\pi}{2^{k-2}}\right)}-1)}$$

It is extremely accurate: if k = 9 we have 699 decimal places of precision, and when k = 100 the precision exceeds 10^29. In the first few cases, the precision doubles with each step.

k	Value	Expression
2	$0.\underline{6931}4020583874...$	$\frac{\pi}{4}-2\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)}$
3	$0.\underline{69314718054}778...$	$\frac{5\pi}{16}-\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/2}-1)}$
4	$0.\underline{69314718055994530}...$ Précision = 21 d.	$\frac{7\pi}{16}-\frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/4}-1)}$
5	$0.\underline{6931471805599453094}...$ Précision = 43 d.	$\frac{85\pi}{128}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/8}-1)}$

If you want to obtain a formula that gives the decimals in base 10 one after the other, you can always modify it, even if it means cheating, by replacing the k in the equation with the logarithm in base 2. This linearises the precision.

$$\ln(2) \approx \frac{\pi(2^{k-1}-2^{1-k})}{6(k-1)}-\frac{2}{k-1}\sum_{n=1}^{\infty}\frac{1}{n\left(e^{\left(\frac{\pi n}{2^{k-2}}\right)}-1\right)}$$

We replace k → $log_2(k)$, it then gives a formula decimal by decimal starting from 4.

As with the other formula, the series on the right converges very slowly but there is a formula that Euler found about pentagonal numbers that will be very useful. First the series

$$\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)}$$

is equivalent to the generating series of partitions (evaluated at $e^{\pi n}$), in fact the following Euler product :

$$\prod_{k\geq 1}^{\infty}\frac{1}{1-x^k}=\sum_{n=0}^{\infty}p(n)\,x^n$$

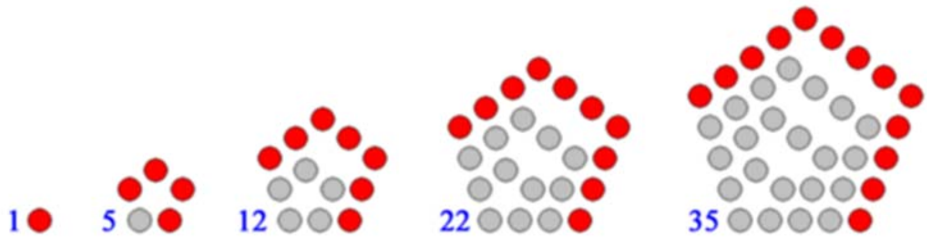
When evaluated in $x = e^{\pi n}$, and when we invert this function

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{k=-\infty}^{k=+\infty}(-1)^k x^{k(3k-1)/2}$$

We find the pentagonal numbers.

$$(1 - x)(1 - x^2)(1 - x^3) \ldots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{26} - \cdots$$

This name comes from the fact that placed on the plane with points they form a pentagon (image from Wikipedia).



So, by inverting the formula with pentagonal numbers (sequence A001318 of the OEIS) and evaluating in $e^{\pi n/m}$we can sum the series whose exponents grow as 3/8 n^2.
The series converges much more quickly, in fact for a precision of 1430652 decimal places when we take d = 20 in the original formula, we need to sum 856159 terms. This gives an efficiency of 869000/1430652, or approximately 0.598. This is known as the Lehmer measure. By way of comparison, the best known formula for calculating ln(2), which is used by the Y-Cruncher (the most efficient known), is the Machin-type formula:

$$\ln(2) = 18\ \mathrm{ArcCoth}(26)\ -\ 2\ \mathrm{ArcCoth}(4801)\ +\ 8\ \mathrm{ArcCoth}(8749)$$

Recall: arccoth(x) = 1/2 ln(x + 1) - 1/2 ln(x - 1), which is equivalent to evaluating the logarithm with an argument close to 1. The higher the value of x, the faster the convergence. This is the most efficient formula known. Its efficiency (Lehmer scale) is 0.616. However, it is not necessary to calculate π. The current record is 1500 billion decimal places (2021) [15].
Another possibility is to find a way of explicitly obtaining the expression of the term (equation 1).

In a previous article, I gave the explicit forms of series with n^k on the numerator.

$S(5,1)$	$\frac{3}{64}\frac{\pi^3}{\Gamma(3/4)^{12}}+\frac{1}{504}$	$\frac{120}{\pi^6}$
$S(5,2)$	$\frac{1}{504}$	$\frac{15}{8\,\pi^6}$
$S(5,4)$	$\frac{-3}{2^{12}}\frac{\pi^3}{\Gamma(3/4)^{12}}+\frac{1}{504}$	$\frac{15}{2^9\,\pi^6}$

Based on this model, using the program Pari-GP (lindep) or PSLQ (Maple version) we can identify the exact expression for S(5,3) and even S(5,a/b). Indeed, for S(5,3) we obtain (based on the same model).

$$S(5,3) = \frac{61}{5184}+\frac{17\sqrt{3}}{2592}-\sqrt{\frac{(154512+89222\sqrt{3})}{23328}\frac{\pi^3}{\Gamma(3/4)^{12}}+\frac{1}{504}}$$

And from there, certain fractional values like

$$S\left(5,\frac{1}{5}\right) = \frac{11547}{64}+\frac{1296\sqrt{5}}{16}+\sqrt{\frac{(1974320\ +882969\sqrt{5})}{16}\frac{\pi^3}{\Gamma(3/4)^{12}}+\frac{1}{504}}$$

This gives some hope of being able to calculate the value of $S\left(-5,\frac{1}{5}\right)$ explicitly. Here the model used above does not work because (mainly) the values of the negative arguments of the function S(-n,a/b) are not simple. For example, $S\left(-1,\frac{1}{2}\right)$is known explicitly:

$$S\left(-1,\frac{1}{2}\right) = \frac{5}{16}\ln(2)-\frac{1}{4}\ln(\sqrt{2}-1)-\frac{1}{4}\ln(\pi)+ln\Gamma\left(\frac{3}{4}\right)-\frac{\pi}{48}$$

The degree of the algebraic term is very unpredictable and so are the coefficients. We do have an expression with (always the same): $ln(2), ln(\pi), \pi$ and $ln\Gamma(3/4)$, the last part is unknown and probably of high degree, so difficult to detect.

These approximations, some of which are spectacular do not stop there,
stop there, combining terms from the Eisenstein series we obtain for example, a function that gives individual positions of $1/\pi^{2n+1}$.

Here [ ] is the integer part.

If m = 2, the precision exceeds 532 digits, if m = 9, the precision exceeds 5 billion digits but if we want individual positions there is a way to construct a function like.

$$\left[\frac{3}{8}\sum_{n=1}^{\infty}\frac{\sigma_1(n)\,n^3}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank 5m-1 of the number 1/π.

$$\left[\frac{1}{8}\sum_{n=1}^{\infty}\frac{\sigma_3(n)\,n^3}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank $7m$ of the number $1/\pi^3$.

This is related to sequence A282213 in the OEIS catalogue, the numbers σ_3 (n) n^3 are the coefficients of the sum of the Eisenstein series. E_2, E_4 and E_6.

$$\left[\frac{1}{384}\sum_{n=1}^{\infty}\frac{\sigma_5(n)\,n^5}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank $11m + 1$ of the number $1/\pi^5$.

$$\left[\frac{1}{9225216}\sum_{n=1}^{\infty}\frac{\sigma_7(n)\,n^7}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank $15m$ of the number $1/\pi^7$.

$$\left[\frac{1}{684343296}\sum_{n=1}^{\infty}\frac{\sigma_9(n)\,n^9}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank g $19m + 2$ of the number $1/\pi^9$.

$$\left[\frac{1}{121639599341568}\sum_{n=1}^{\infty}\frac{\sigma_{11}(n)\,n^{11}}{e^{\left(\frac{\pi n}{10^m}\right)}}\right] mod\ 10$$

Give the decimal digit of rank $23m + 1$ of the number $1/\pi^{11}$.

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